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1997 J. Phys. A: Math. Gen. 30 927

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Is the level spacing distribution of the infinite-dimensional harmonic oscillator that of a Poisson process?

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Received 9 April 1996

Abstract. By averaging over the frequencies of the oscillations, the spacing distribution of the infinite-dimensional harmonic oscillator is shown to be e^{-s} , following the universal behaviour of spacing distributions of integrable systems. For oscillators of finite dimension, n , reasons for the model distribution, $\zeta(n)^{-1}e^{\zeta(n)^{-1}s}$, are suggested. Numerical results follow.

1. Introduction

At the classical level, the defining equations of a dynamical system can be linear or non-linear. This gives rise to properties of integrability and chaos, with mixed KAM states between. However, when the same system is studied at the quantum level, the Schrodinger equation is applied, which is linear.

The question of how the nonlinear properties of the classical system are absorbed into the quantum mechanics need to be addressed. Similarly, properties of the quantum system that give information about the classical system are worth investigating.

For a bounded system, the quantized energies take discrete values. If the spacings between consecutive energy values are normalized to have an average of unity, then a probability distribution can be defined from these spacings. This is known as the level spacing distribution.

For a system that is classically chaotic, the level spacing distribution is the same as that obtained from the spacings between eigenvalues of certain random matrices. For a system with time-reversal invariance, the Gaussian orthogonal ensemble of real symmetric matrices is obtained (the levels show linear repulsion). For a system without the invariance, the Gaussian unitary ensemble of real complex matrices is found (the levels have quadratic repulsion). These have been found for numerous examples, see [1–3] for a few.

For classically integrable systems, it has been observed and shown by Berry and Tabor [4] that the statistics follow those of a Poisson process (the levels are independent).

Thus the level spacing distributions are split into different universality classes, depending upon the originating systems classical chaoticity; it is a signature for chaos.

The proof in [4] for classically integrable systems fails for one important class of systems, however. These are the harmonic oscillators, which is a standard model for various systems, ranging from polyatomic molecules to the phonon spectrum of solids (see [5]). For any integrable system, the energy surface can be transformed into the action-angle coordinates. The flat nature of the energy surface of harmonic oscillators in action space causes the method of stationary phase employed in [4] to break down. Pandey *et al* [5, 6]

showed that the two-dimensional oscillator has an unstable level spacing distribution. In [7], however, an average was formed that removed the instabilities noted by Pandey *et al* and the following result obtained for generic frequency ratio

$$P(s) = \begin{cases} \zeta(2)^{-1} & s \leq 1 \\ \zeta(2)^{-1} \left[\frac{1}{s} + 2 \left(\frac{s-1}{s} \right)^2 \log \left(\frac{s-1}{s} \right) - \frac{1}{2} \left(\frac{s-2}{s} \right)^2 \log \left| \frac{s-2}{s} \right| \right] & s \geq 1. \end{cases} \quad (1)$$

This method also enabled $P(s)$ to be determined for specific values of frequency ratio. However, the method relied heavily on continued fraction theory and so related to the two-dimensional case only. The following work averages over the frequencies of the oscillations to obtain $P(s)$ for harmonic oscillators of general dimension.

The following results will be proved.

Theorem 1. For the n -dimensional harmonic oscillator (n DHO), provided that s is small, the average level spacing distribution is approximately,

$$P(s) = \sum_{r=0}^{n-2} \zeta(n)^{-r-1} \frac{s^r}{r!} \quad (2)$$

where $\zeta(n)$ is the Reimann zeta function.

As the dimension of the oscillator becomes higher, the approximation will become more exact, ultimately yielding,

Corollary 1.1. For the infinite-dimensional harmonic oscillator, the average level spacing distribution is that of a Poisson process, i.e. e^{-s} , following the universal behaviour associated with integrable systems.

Note that the finite-dimensional harmonic oscillator can be approximated quite closely by the function $\zeta(n)^{-1} e^{-\frac{s}{\zeta(n)}}$. Although the total probability of this function is unity, this is unlikely to be the exact level spacing distribution, as the average is $\zeta(n)^{-1}$, and not unity.

2. Averaging over the frequencies of the oscillations

For any physical system there exists the Hamiltonian $H(\mathbf{p}, \mathbf{q})$ which is dependent upon $2n$ momentum and position coordinates, respectively, where n is the dimension of the system. For integrable systems, the phase space (\mathbf{p}, \mathbf{q}) can be transformed into action-angle coordinates $(\mathbf{I}, \boldsymbol{\alpha})$, where the Hamiltonian depends only upon the action coordinates \mathbf{I} . If $\mathbf{I} = \hbar \mathbf{m} + \boldsymbol{\gamma}$ where \mathbf{m} is any vector in the unit lattice (with positive coordinates) and $\boldsymbol{\gamma}$ is a constant vector, known as the Maslov index, then $E_m = H(\mathbf{I})$ give the discrete values of energy of the quantized system. This method of quantization is known as the EBK quantization. More details of these methods may be found in [2] or [4].

As the differences between energy levels are used to obtain the level spacing distribution, $\boldsymbol{\gamma}$ is irrelevant and will be taken as zero. For an n -dimensional harmonic oscillator, the Hamiltonian is,

$$H(\mathbf{I}) = \mathbf{I} \cdot \boldsymbol{\omega}$$

where ω_i are the frequencies of the oscillators. If U denotes the volume enclosed by the energy surface in \mathbf{m} space then,

$$U(\mathbf{m}) = \frac{(\boldsymbol{\omega} \cdot \mathbf{m})^n}{n! \prod \omega_i}.$$

If $\alpha_i = \omega_i/\omega_n$ then this becomes

$$U(\mathbf{m}) = \frac{(\boldsymbol{\alpha} \cdot \mathbf{m})^n}{n! \prod \alpha_i} \tag{3}$$

where $\alpha_n = 1$. Let the surface initially intersect \mathbf{m} , then increase U to U' where the next lattice point, \mathbf{m}' , is intersected. Degeneracy among the energy levels corresponds to commensurate ratios, α_i , whereupon zero spacings ultimately contribute to give $\delta(s)$ as the spacing distribution. On average, degeneracy among the energies does not occur, so will be ignored in what follows. The shift vector, \mathbf{p} , is defined by,

$$\mathbf{p} = \mathbf{m}' - \mathbf{m}. \tag{4}$$

Now the spacing is the difference between consecutive volumes, so

$$\begin{aligned} s = U' - U &= \frac{(\boldsymbol{\alpha} \cdot \mathbf{m}')^n - (\boldsymbol{\alpha} \cdot \mathbf{m})^n}{n! \prod_{i=1}^n \alpha_i} \quad (\text{by (3)}) \\ &= \frac{\boldsymbol{\alpha}(\mathbf{m}' - \mathbf{m})}{n! \prod_{i=1}^n \alpha_i} [(\boldsymbol{\alpha} \cdot \mathbf{m}')^{n-1} + (\boldsymbol{\alpha} \cdot \mathbf{m}')^{n-2} \boldsymbol{\alpha} \cdot \mathbf{m} + \dots + (\boldsymbol{\alpha} \cdot \mathbf{m})^{n-1}] \\ &\approx \frac{\boldsymbol{\alpha} \cdot \mathbf{p}}{n! \prod_{i=1}^n \alpha_i} n(\boldsymbol{\alpha} \cdot \mathbf{m})^{n-1} \quad (\text{by (4)}) \\ &= \frac{\boldsymbol{\alpha} \cdot \mathbf{p}}{(n-1)! \prod_{i=1}^n \alpha_i} (\boldsymbol{\alpha} \cdot \mathbf{m})^{n-1}. \end{aligned} \tag{5}$$

To obtain $P(s)$, the following method is employed. First $\boldsymbol{\alpha}$ is randomly selected from some region with volume V (this will also be denoted as region V). The region V must be small to ensure that the resulting spacing distributions have approximately the same average spacing. Note that V has dimension $n-1$, as $\alpha_n = 1$. Consider the subregion of the volume V , such that the spacing induced is under some uppermost limit, λ , for some shift vector, \mathbf{p} . Then by (5), the following inequality must be satisfied,

$$0 < \boldsymbol{\alpha} \cdot \mathbf{p} < \frac{(n-1)! \prod_{i=1}^n \alpha_i \lambda}{(\boldsymbol{\alpha} \cdot \mathbf{m})^{n-1}} = c_\lambda. \tag{6}$$

Note that the coordinates of the shift vector, \mathbf{p} , must be coprime. If this is not the case, then by dividing through with the highest common divisor, a new shift vector will be obtained which will induce a smaller spacing than the original. As a semi-classical analysis, it is assumed that the quantum numbers are quite large, so $\boldsymbol{\alpha} \cdot \mathbf{m}$ is assumed to be of large order. Thus as $\boldsymbol{\alpha}$ varies over the volume V , the main variation in equality (6) arises in the variation of $\boldsymbol{\alpha} \cdot \mathbf{p}$, so c_λ is taken as constant.

Thus there is a subvolume V_p in V , such that if $\boldsymbol{\alpha}$ lies in V_p , then the shift vector that gives the spacing is \mathbf{p} . Then if the probability that a randomly chosen spacing is less than λ is $P(s < \lambda)$,

$$P(s < \lambda) = \frac{1}{V} \sum_{\mathbf{p}} V_p. \tag{7}$$

The three-dimensional case is outlined in figure 1. Note that only lattice points in the region, R , indicated, will result in a shift vector, \mathbf{p} , that contributes to the sum in (7).

The average level spacing distribution is then finally obtained from (7) via

$$P(\lambda) = \frac{d}{d\lambda} P(s < \lambda). \tag{8}$$

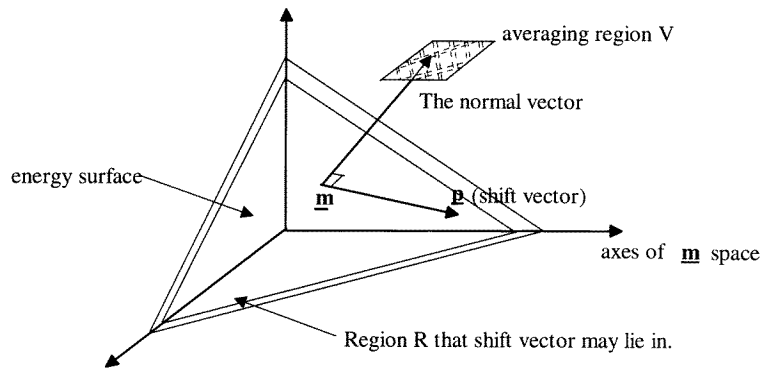


Figure 1. The energy surface in m space.

3. The main result

The inequality of (6) defines a narrow strip in α space of width $c_\lambda |p|^{-1}$. If p_n is given free rein over the integers, a series of parallel strips is obtained because $\alpha_n = 1$. These are a perpendicular distance $|p|^{-1}$ apart. If $q = (p_1, \dots, p_{n-1})$, let W_q denote the intersection of V with the union of all the strips, and $|W_q|$ the volume of this intersection. If $\bar{V}_q = \sum_{p_n} V_p$, then (7) becomes,

$$P(s < \lambda) = \frac{1}{V} \sum_q \bar{V}_q. \tag{9}$$

Now $|W_q|$ represents a good estimate of \bar{V}_q . However, if q and q_2 are distinct, and $\alpha \in W_q \cap W_{q_2}$, then the shift vector that corresponds to the spacing for that value of α , could derive from q or q_2 . On average, half the region $W_q \cap W_{q_2}$ will give a shift vector derived from q . Thus, $|W_q| - \frac{1}{2} \sum_{q_2} |W_q \cap W_{q_2}|$ is a better estimate of \bar{V}_q . However, for any q_3 distinct from q and q_2 , when $\alpha \in W_q \cap W_{q_2} \cap W_{q_3}$, the shift vector will correspond to q , on average, for a third of the volume $|W_q \cap W_{q_2} \cap W_{q_3}|$. Thus, for each q_2, q_3 , the volume $\frac{1}{2} |W_q \cap W_{q_2}|$ taken off at the previous step, removed, on average, a volume $\frac{1}{3} |W_q \cap W_{q_2} \cap W_{q_3}|$ that would give a shift vector corresponding to q . So a volume $\frac{1}{3!} |W_q \cap W_{q_2} \cap W_{q_3}|$ must be added back on.

This process of alternately removing and replacing smaller and smaller terms continues until,

$$\begin{aligned} \bar{V}_q &= |W_q| - \frac{1}{2} \sum_{q_2} |W_q \cap W_{q_2}| + \frac{1}{3!} \sum_{q_2, q_3} |W_q \cap W_{q_2} \cap W_{q_3}| - \dots + \frac{(-1)^n}{(n-1)!} \\ &\times \sum_{q_2, \dots, q_{n-1}} |W_q \cap \dots \cap W_{q_{n-1}}|. \end{aligned} \tag{10}$$

As V is an $(n-1)$ -dimensional space, for q, q_2, \dots, q_n distinct, $W_q \cap W_{q_2} \cap \dots \cap W_{q_n}$ will in general be an empty set, so make no contributions to the above sum, and the process terminates. However, this assumption of no intersection will break down once the width of the strips $(c_\lambda/|p|)$ become large enough. This problem is investigated in section 4.

Thus the volumes $|W_{q_1} \cap W_{q_2} \cap \dots \cap W_{q_i}|$ need to be determined. Now the set of strips W_{q_i} are described by $q_i \cdot \alpha = \beta_i$. Considering $|W_{q_1} \cap W_{q_2} \cap \dots \cap W_{q_{n-1}}|$ first,

$$\begin{aligned} q_1 \cdot \alpha &= \beta_1 \\ q_2 \cdot \alpha &= \beta_2 \\ &\vdots \\ q_{n-1} \cdot \alpha &= \beta_{n-1} \end{aligned} \tag{11}$$

where β_i range over an interval of length c_λ (by (6)). Then define the matrix $Q^T = (q_1 \cdot q_2 \cdot \dots \cdot q_{n-1})$ to give $Q\alpha = \beta$. In general, Q will have an inverse (see section 4). Putting $x = Q\alpha$ yields $x = \beta$. As each β_i varies over an interval of length c_λ , each parallelepiped of intersection has volume c_λ^{n-1} in x space, so volume $c_\lambda^{n-1}/|Q|$ in α space.

In x space, the corresponding planes obtained from transforming the planes of (11) in α space, are perpendicular, and for consecutive values, p_n , a perpendicular distance of unity apart. Thus the parallelepipeds occur with density unity in x space. The volume V averaged over in α space becomes $V|Q|$ in x space, thus,

$$|W_{q_1} \cap W_{q_2} \cap \dots \cap W_{q_{n-1}}| = c_\lambda^{n-1} V.$$

If the region $W_{q_1} \cap W_{q_2} \cap \dots \cap W_{q_i}$ is being considered, then this corresponds to the intersection of the planes,

$$\begin{aligned} q_1 \cdot \alpha &= \beta_1 \\ &\vdots \\ q_i \cdot \alpha &= \beta_i. \end{aligned} \tag{12}$$

The same transformation as above can be applied. The intersection then becomes an i -dimensional parallelepiped of cross-section c_λ^i in x space, with the other $n-1-i$ dimensions forming a ‘tube’, perpendicular to the cross-section, running through the surfaces as defined in (12). As the surfaces embedded in x space are a unit distance apart, the $(n-1-i)$ -dimensional ‘tubes’ will have length (i.e. an $(n-1-i)$ -dimensional volume) of $V|Q|$, so,

$$|W_{q_1} \cap W_{q_2} \cap \dots \cap W_{q_i}| = c_\lambda^i V. \tag{13}$$

Then using (10) and (13), yields,

$$\bar{V}_q = c_\lambda V - \frac{1}{2} \sum_{q_2} c_\lambda^2 V + \frac{1}{3!} \sum_{q_2, q_3} c_\lambda^3 V - \dots + \frac{(-1)^n}{(n-1)!} \sum_{q_2, \dots, q_{n-1}} c_\lambda^{n-1} V$$

so substituting into (8),

$$P(s < \lambda) = \sum_q c_\lambda - \frac{1}{2} \sum_{q, q_2} c_\lambda^2 + \frac{1}{3!} \sum_{q, q_2, q_3} c_\lambda^3 - \dots + \frac{(-1)^n}{(n-1)!} \sum_{q, q_2, \dots, q_{n-1}} c_\lambda^{n-1}. \tag{14}$$

Let $\{N\}$ denote the set of possible vectors, q , that give rise to a shift vector, and N denote the size of this set. Then,

$$\sum_{q, \dots, q_i} c_\lambda^i = c_\lambda^i N^i. \tag{15}$$

But N will just be the product of the surface area of the energy surface, resolved in direction p_n , with a factor, $\zeta(n)^{-1}$, accounting for the coprime condition. Then let A denote the surface area, and ϑ be the angle between p_n and the surface normal vector α .

Then $N = A \cos \vartheta$. First evaluate the surface area. From (3), the energy surface is defined by,

$$U(\mathbf{m}) = \frac{(\boldsymbol{\alpha} \cdot \mathbf{m})^n}{n! \prod_{i=1}^n \alpha_i}$$

or equivalently,

$$\boldsymbol{\alpha} \cdot \mathbf{m} = \sqrt[n]{Un! \prod_{i=1}^n \alpha_i}.$$

Denote the perpendicular distance of the surface from the origin (of \mathbf{m} space) by r , then

$$r = \frac{\sqrt[n]{Un! \prod_{i=1}^n \alpha_i}}{|\boldsymbol{\alpha}|}.$$

Then

$$A = \frac{dU}{dr}$$

which yields,

$$A = \frac{|\boldsymbol{\alpha}|(\mathbf{m} \cdot \boldsymbol{\alpha})^{n-1}}{(n-1)! \prod_{i=1}^n \alpha_i}.$$

Also,

$$\cos \vartheta = \frac{\boldsymbol{\alpha} \cdot \mathbf{p}_n}{|\boldsymbol{\alpha}|p_n} = \frac{1}{|\boldsymbol{\alpha}|} \quad (\text{as } \alpha_n = 1)$$

which together give,

$$A \cos \vartheta = \frac{(\mathbf{m} \cdot \boldsymbol{\alpha})^{n-1}}{(n-1)! \prod_{i=1}^n \alpha_i}. \quad (16)$$

Thus finally, using (6),

$$c_\lambda N = c_\lambda \zeta(n)^{-1} A \cos \vartheta = \zeta(n)^{-1} \frac{(n-1)! \prod_{i=1}^n \alpha_i \lambda}{(\mathbf{m} \cdot \boldsymbol{\alpha})^{n-1}} \frac{(\mathbf{m} \cdot \boldsymbol{\alpha})^{n-1}}{(n-1)! \prod_{i=1}^n \alpha_i} = \zeta(n)^{-1} \lambda.$$

Thus (15) gives,

$$\sum_{q_1, \dots, q_i} c_\lambda^i = \zeta(n)^{-i} \lambda^i$$

which combining with (14) gives,

$$P(s < \lambda) = \sum_{i=1}^{n-1} \frac{\zeta(n)^{-i} \lambda^i}{i!}. \quad (17)$$

Then using (8) to obtain the probability distribution gives,

$$P(\lambda) = \zeta(n)^{-1} \sum_{i=0}^{n-2} \frac{1}{i!} \left(\frac{\lambda}{\zeta(n)} \right)^i$$

which is the distribution given in theorem 1. As $n \rightarrow \infty$ note that $\zeta(n) \rightarrow 1$, so the distribution given in corollary 1.1 is obtained,

$$P(\lambda) = e^{-\lambda}$$

as conjectured by Pandey *et al* [5, 6].

As pointed out above, these expressions only remain valid provided that c_λ is sufficiently small. The range of validity is investigated in the next section.

4. The range of validity

In section 3 various volumes were estimated in the calculation of $P(S < \lambda)$ in (14). There were two main assumptions involved in the counting of these volumes, however. This section addresses those assumptions in turn.

The first assumption made was that the intersections $W_q \cap W_{q_2} \cap \dots \cap W_{q_n}$ of (10) were transversal, or equivalently, that the matrix Q was invertible. This will not always be the case. However, we have the following result.

Lemma 1. The proportion of possible matrices Q with determinant zero, tends to zero in the semi-classical limit $m \rightarrow \infty$.

Proof. Assume that $W_q \cap W_{q_2} \cap \dots \cap W_{q_{i-1}}$ is a transversal intersection, then consider the set of possible vectors, q_i , that give rise to a region W_{q_i} that does not intersect $W_q \cap W_{q_2} \cap \dots \cap W_{q_{i-1}}$ transversally. The set of possible vectors, q , was defined after (14) as $\{N\}$, where it was found in (16) that

$$N = A \cos \vartheta = \frac{(m \cdot \alpha)^{n-1}}{(n-1)! \prod_{i=1}^n \alpha_i}.$$

The ‘non-transversal’ vector q_i will be taken from this set. Let $\{q, \dots, q_{i-1}\}$ denote the linear basis formed by vectors q, \dots, q_{i-1} . Vectors in this basis with integer coordinates are the non-transversal ones required. The volume of intersection of this basis with $\{N\}$ will be an upper bound on the number of non-transversal vector q_i . This volume will be of the order of $\frac{(m \cdot \alpha)^{i-1}}{(i-1)!}$. Thus the proportion of vectors giving nontransversal intersections is of the order of $(m \cdot \alpha)^{i-1-n} \frac{(n-1)!}{(i-1)!}$. As the dimension n is fixed, and the analysis semi-classical, which means $|m|$ is taken as being arbitrarily large, the proportion of non-transversal vectors is arbitrarily close to zero, which completes the proof. \square

The second assumption made is that as V is an $(n - 1)$ -dimensional space, $W_q \cap W_{q_2} \cap \dots \cap W_{q_n}$ will be an empty set. This is not necessarily the case. By lemma 1, the intersection $W_q \cap W_{q_2} \cap \dots \cap W_{q_{n-1}}$ is transversal and the matrix Q is invertible. The region $W_q \cap \dots \cap W_{q_{n-1}}$ is described by the simultaneous equations of (11). Then for any vector q_n (again W_{q_n} is taken as transversal) we have

Lemma 2. The intersection $W_q \cap W_{q_2} \cap \dots \cap W_{q_{n-1}} \cap W_{q_n}$ is an empty set in the limit $n \rightarrow \infty$.

Proof. From (11) the intersection $W_q \cap W_{q_2} \cap \dots \cap W_{q_{n-1}}$ is described by the simultaneous equations

$$\begin{aligned} q \cdot \alpha &= \beta_i \\ &\vdots \\ q_{n-1} \cdot \alpha &= \beta_{n-1}. \end{aligned}$$

Where β_i are (very close to) integers. Under the transformation $x = Q\alpha$, this becomes $x_i = \beta_i$. The region W_{q_n} is defined by $q_n \cdot \alpha = \beta_n$. As Q is assumed to be invertible, in x space we obtain

$$q_n \cdot Q^{-1}x = \beta_n \Leftrightarrow x \cdot (Q^{-1})^T q_n = \beta_n.$$

If R_{ij} denotes the matrix of cofactors of Q then

$$x_i R_{ij}(q_n)_j = \beta_n |Q| \Leftrightarrow x \cdot Rq_n = \beta_n |Q|. \tag{18}$$

Then as the dimension $n \rightarrow \infty$, $|Q|$ becomes very large, and the highest common factor of the elements $R_{ij}(q_n)_j$ tend to unity. Thus the proportion of vectors x (with integer

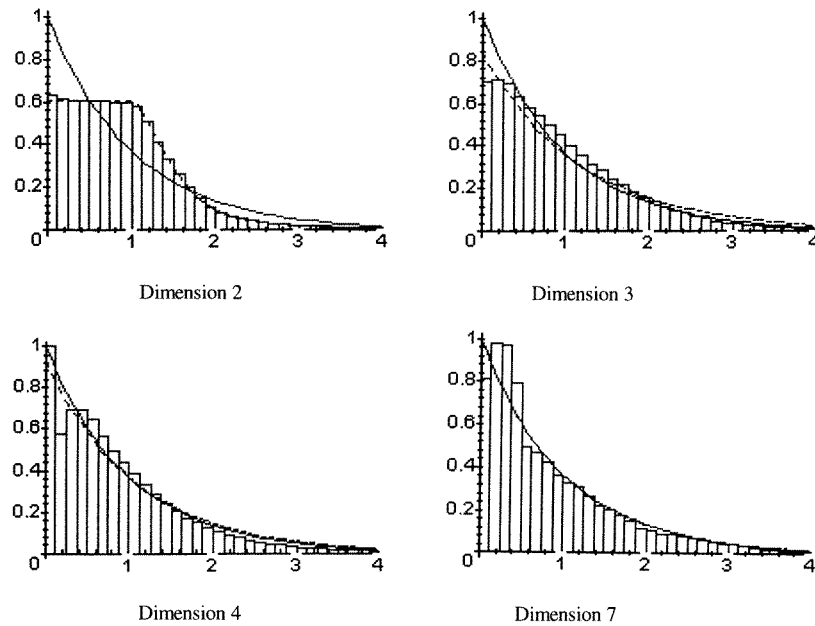


Figure 2. Averaged spacing distributions for various harmonic oscillators.

coordinates) that are solutions to (18) is $|Q|^{-1}$, which tends to zero as the dimension becomes infinite.

It has been assumed that the width of the strips c_λ is negligible. However, if λ is large enough, this is not the case. In the limit of infinite dimension, λ , can be arbitrarily large, and the number of lattice points, α , that (18) is satisfied by (as the range of β_n increases) will still be of negligible proportion. This completes the proof. \square

5. Numerical results

It has been shown that the average distribution for the infinite-dimensional harmonic oscillator will be the exponential distribution of the Poisson process. How close the model function $\zeta(n)^{-1}e^{-\zeta(n)^{-1}s}$ represents the actual distribution, was numerically investigated (see figure 2).

To obtain the distributions shown in the figure the spacings from the first 100 000 levels were obtained for 10 000 different frequency ratios. This took about three hours on a VAX mainframe computer. The exponential distribution e^{-s} associated with integrable systems is indicated by a curve in all four cases.

For the two-dimensional oscillator, distribution (1) was rigorously derived in [7], so the distribution $\zeta(2)^{-1}e^{-\zeta(2)^{-1}s}$ will not be a good approximation. However, the numerics can be seen to be a very close fit to (1), which is indicated by the broken curve. It was also observed that a good approximation to (1) could be obtained by quicker numerics. For instance, (1) readily presents itself if the distribution from 1000 levels is averaged over 1000 frequency ratios.

For the three-dimensional oscillator, the numerics can be seen to fit the model distribution $\zeta(3)^{-1}e^{-\zeta(3)^{-1}s}$ (indicated by the broken curve) quite closely. Similar statistics were found in other cases, with good agreement for small s , rising above the function for

$\frac{1}{2} < s < 2$, and then dipping below.

For dimensions above three, the model distribution $\zeta(n)^{-1}e^{-\zeta(n)^{-1}s}$ is very close to the exponential distribution e^{-s} , as are the numerics. The statistics fluctuated a lot more in the higher-dimensional cases, which would seem to suggest a higher degree of averaging was required. This may be due to the following observation. In dimension two, one frequency ratio, a_1 , was averaged over 10 000 values. In the three-dimensional case, each frequency ratio, a_1, a_2 , was averaged over 100 values. In the n -dimensional case, each frequency ratio, a_1, \dots, a_{n-1} , is averaging over $10\,000^{\frac{1}{n}}$ different values. To get the consistency of the results of the two-dimensional case, it would suggest averaging over $10\,000^n$ different frequencies, making the infinite-dimensional case difficult to observe. However, as the dimension increased, the statistics rapidly approached those of a Poisson process.

6. Conclusions

The average level spacing distributions for harmonic oscillators in general dimension have been investigated. For the infinite dimensional oscillator, the distribution was found to follow the universal behaviour associated with integrable systems, i.e. that of a Poisson process, $P(s) = e^{-s}$. For finite-dimensional oscillators (above dimension two), the distribution was well approximated by the function $\zeta(n)^{-1}e^{-\zeta(n)^{-1}s}$, which gives a total probability of unity and a mean that is approximately unity. Note that the distribution becomes exact in the limit $s \rightarrow 0$.

Considering the two-dimensional case in particular, it was interesting to see distribution (1) reflected so well in the numerics. The method of averaging over the frequencies used in this paper, is less powerful than that used in [7], where it was possible to obtain an analytic expression for the spacing distribution, for any particular frequency ratio, using its continued fraction. However, that method was not easily extended to higher dimensions, due to its use of continued fraction theory, which is readily applicable in two dimensions. It may be possible to extend those results into higher dimension, using Jacobi–Perron theory [8, 9] which would be an interesting further study. One would hope to obtain the average distribution derived above for small spacings, along with a number theoretical correction, corresponding to the larger spacings, much as in the two-dimensional case.

As to an answer to the title of the paper, it would seem that the infinite-dimensional harmonic oscillator does give an exponential spacing distribution which, although difficult to observe, is rapidly approximated by finite-dimensional oscillators.

Acknowledgments

The author would like to thank Sandy Davie for many helpful comments. Points made by Professor M Berry and those on the referees report were also useful.

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